DISCRETE EVENT DYNAMIC SYSTEMS

STOCHASTIC TIMED AUTOMATA AND PETRI NETS

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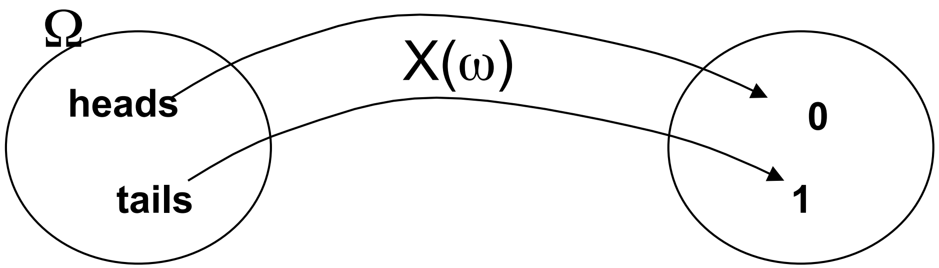
STOCHASTIC TIMED DES

- Uncertainty arises in DES from unpredictable human actions, machine failures, sensor noise, etc
- The Untimed view accounted for "*all possible behaviors*", but was limited to logical or *qualitative* performance analysis
- The Deterministic Timed view shed some light on the *quantitative* performance regarding the dynamic behavior of DES, but deterministic clock structures only capture a single timed string of events (or states)
- We will now incorporate models of uncertainty that involve stochastic elements, so as to analyze *quantitative* performance and search for optimal behaviors
- The main effort will be directed towards developing a *stochastic clock structure* model for the input of our timed automata

BRIEF REVIEW OF STOCHASTIC PROCESSES

- Definition: A *stochastic* or *random process* $X(\omega, t)$ is a collection of random variables indexed by t. The random variables are defined over a common probability space (Ω, E, P) with $\omega \in \Omega$. The variable t ranges over some given set $T \subseteq R$.

  if ω is fixed, $X(\omega,t)$ is a deterministic time function called *sample path* or *realization* of the SP.



if t is fixed, $X(\omega,t)$ is a *random variable.*

$X(\omega,1), X(\omega,2), X(\omega,3), X(\omega,4), X(\omega,5),\ldots=0,1,1,0,0$ ⟶ *sample path*

To simplify the notation, we will denote a SP by {X(t)} or *X(t)*

BRIEF REVIEW OF STOCHASTIC PROCESSES
Classification concerning state space and time

DISCRETE-STATE PROCESSES (or CHAINS): when the state space is defined over a finite or countable set
CONTINUOUS-STATE PROCESSES: in the other cases

DISCRETE-TIME PROCESSES: when the index variable *t* is defined over a finite or countable set – STOCHASTIC or RANDOM *SEQUENCE* X_k
CONTINUOUS-TIME PROCESSES: in the other cases

BRIEF REVIEW OF STOCHASTIC PROCESSES
Classification concerning statistical characteristics

To fully characterize a stochastic process we need to specify the joint cumulative distribution function (cdf).

Given a random vector:

$$X = \left[X(t_0), X(t_1), \ldots, X(t_n)\right]$$

which can take values

$$\mathbf{x} = (x_0, \ldots, x_n)$$

the joint cumulative distribution function (cdf) is given by

$$F_X(x_0, \ldots, x_n; t_0, \ldots, t_n) = P(X(t_0) \le x_0, \ldots, X(t_n) \le x_n)$$

BRIEF REVIEW OF STOCHASTIC PROCESSES
Classification concerning statistical characteristics

STATIONARY PROCESSES
(strict sense)

$$F_X(x_0,\ldots,x_n;t_0,\ldots,t_n) = F_X(x_0,\ldots,x_n;t_0+\tau,\ldots,t_n+\tau) \quad \forall \tau \in \Re$$

wide sense stationary process IFF

$E[X(t)]=C$ and $E[X(t)X(t+\tau)]=g(\tau)$

independent process IFF

$$F_X(x_0,\ldots,x_n;t_0,\ldots,t_n) = F_{X_0}(x_0,t_0)F_{X_1}(x_1,t_1)\ldots F_{X_n}(x_n,t_n)$$

if all random variables are drawn from the same distribution we call it *independent and identically distributed* (*iid*)

MARKOV PROCESSES

If the future states of a process are conditionally independent of the past history, given the present state *(memoryless property),* i.e.,

$$P[X(t_{k+1}) \leq x_{k+1} | X(t_k) = x_k, X(t_{k-1}) = x_{k-1}, \ldots, X(t_0) = x_0] =$$
$$P[X(t_{k+1}) \leq x_{k+1} | X(t_k) = x_k)]$$

the SP has the *Markov property* and is a *Markov process*.

For *Markov Chains* (discrete state space) in discrete-time:

$$P[X_{k+1} = x_{k+1} | X_k = x_k, X_{k-1} = x_{k-1}, \ldots, X_0 = x_0) = P(X_{k+1} = x_{k+1} | X_k = x_k)]$$

MARKOV PROCESSES

Memoryless property:

(M1) All past state information is irrelevant (no state memory needed)

(M2) How long the process has been in the current state is irrelevant (no state age memory needed)

SEMI-MARKOV PROCESSES

Constraint **(M2)** is relaxed

RENEWAL PROCESS – is a chain {*N(t)*} with state space {0,1,2,...} whose purpose is to count state transitions. The time intervals defined by successive state transitions are assumed to be independent and characterized by a common distribution, which may be arbitrary. Normally, $N(0)=0$. $N(0) \leq N(t_1) \leq \ldots N(t_k)$

For DES, *a renewal process counts the number of events that occur in the time interval]0,t]*

STOCHASTIC TIMED AUTOMATA
Stochastic Clock Structures

The ***Stochastic Clock Structures*** (or *Timing Structure*) associated with an event set *E* is a set of distribution functions

$$G = \{G_i : i \in E\}$$

characterizing the *stochastic clock sequences*

$$V_i = \{v_{i,1}, v_{i,2}, \ldots\}, i \in \xi, v_{i,k} \in \Re^+, k = 1,2,\ldots$$

We will be concerned with iid clock sequences, which are independent of each other. Therefore, each $\{v_{i,k}\}$ is completely characterized by a distribution function $G_i(t)=P[v_i \leq t]$

STOCHASTIC TIMED AUTOMATON

Def.: A **Stochastic Timed Automaton** (STA) is a six-tuple:

$$(\xi, \chi, \Gamma, p, p_0, G)$$

where

ξ is a countable event set

χ is a countable state space

$\Gamma(x)$ is a set of feasible or enabled events, defined for all $x \in \chi$, with $\Gamma(x) \subseteq \xi$

$p(x' \mid x, e')$ is a state transition probability, defined $\forall_{x,x' \in \chi; e' \in \xi}$ and such that $p(x' \mid x, e') = 0 \quad \forall_{e' \notin \Gamma(x)}$

$p_0(x)$ is the probability mass function (pmf) $P[X_0 = x], x \in \chi$, of the initial state X_0

$G = \{G_i : i \in \xi\}$ is a stochastic clock structure

STOCHASTIC TIMED AUTOMATON

The STA generates a stochastic sequence $\{X_0, X_1, ...\}$ through a transition mechanism (based on observations $X=x$, $E'=e'$):

$X'=x'$ with probability $p(x';x,e')$

Driven by a stochastic event sequence $\{E_1,E_2,...\}$ generated through

$$E' = \arg\min_{i\in\Gamma(x)} \{Y_i\}$$

With the stochastic *clock values* Y_i, $i\in\xi$, defined by

$$Y_i' = \begin{cases} Y_i - Y^* & \text{if } (i \neq E') \wedge i \in \Gamma(X) \\ v_{i,N_{i+1}} & \text{if } (i = E') \vee i \notin \Gamma(X) \end{cases} \qquad i \in \Gamma(X')$$

Where the interevent time Y^* is

$$Y^* = \min_{i\in\Gamma(x)} \{Y_i\}$$

STOCHASTIC TIMED AUTOMATON

And *the event scores* N_i, $i \in \xi$, defined by

$$N_i' = \begin{cases} N_i + 1 & \text{if } (i = E') \vee i \notin \Gamma(X) \\ N_i & \text{if } (i \neq E') \wedge i \in \Gamma(X) \end{cases} \qquad i \in \Gamma(X')$$

In addition, event times are updated according to

$$T' = T + Y^*$$

Note that $\{v_{i,k}\} \sim G_i$

Also:

$X_0 \sim p_0(x)$

$Y_i = v_{i,1}$, $N_i = 1$, if $i \in \Gamma(X_0)$

Y_i undefined, $N_i = 0$, if $i \notin \Gamma(X_0)$

STOCHASTIC TIMED AUTOMATON
Generalized Semi-Markov Process

A Generalized Semi-Markov Process is a stochastic process {X(t)} with state space χ, generated by a stochastic timed automaton $(\xi, \chi, \Gamma, p, p_0, G)$.

The Markovian aspect comes from

$X' = x'$ with $p(x'|x,e')$

but note that the *interevent times* Y^* have, in general, arbitrary distributions (SMP).

However, unlike for SMP, the distribution of Y^* is not given, but rather depends on the distributions G_i, and on the *clock* and *score* updating mechanisms (Generalized SMP)

GSMP ANALYSIS

Questions of Interest

P[$X(t)=x$] – probability of finding the DES modeled by the STA at state x at time t.

Interevent time sequence $\{Y^*\} = \{Y_0^*, Y_1^*, Y_2^*,\ldots\}$ and (related) *state holding times*.

Event sequence $\{E_k\} = \{E_1,E_2,\ldots\}$ $\Rightarrow P[E_k=i]$ or $P[E_k=i|X_k=x]$

Score process $N_i(t)$ – counting process for event i occurrences in the interval $]0,t]$

POISSON COUNTING PROCESS

- DES with a single event.
- *N(t)* counts the number of event occurences in]0,t] (state space is {0,1,2,...}) $N(0) \le N(t_1) \le \ldots N(t_k)$ $P_n(t) := P[N(t)=n]$
- Partitioning the time line into an arbitrary number of intervals $]t_{k-1},t_k]$ of arbitrary lengths, as well as setting $t_0=0$ and assuming $N(0)=0$, we define

$$N(t_{k-1},t_k)=N(t_k)-N(t_{k-1}),\ k=1,2,\ldots$$

Counts the number of events occurring in the interval $]t_{k-1},t_k]$

- **Assumptions**

(A1) at most one event can occur at any time instant

(A2) $N(t), N(t,t_1), N(t_1,t_2), N(t_2,t_3),\ldots$ are *mutually independent* for any $0 \le t \le t_1 \le t_2$

(A3) $P[N(t_{k-1},t_k)=n]=P[N(s)=n]$, $s= t_k- t_{k-1}$, is independent of t_{k-1} and t_k but may depend of t_k-t_{k-1} – a form of *stationarity*

Def.: A process that satisfies (A2) is a process with *independent increments*

If (A2)+(A3) are satisfied, it is a process with *stationary independent increments*

POISSON COUNTING PROCESS

By **(A3)**:

P[N(t,t+s)=n]=P[N(s)=n] (*)

P[N(t+s)=0] = P[N(t)=0 and N(t,t+s)=0]

By **(A2)**:

P[N(t+s)=0] = P[N(t)=0].P[N(t,t+s)=0]

from (*) P[N(t+s)=0]= P[N(t)=0].P[N(s)=0]

Using the definition $P_n(t)$:=P[$N(t)=n$]

$$P_0(t+s)= P_0(t).\ P_0(s)$$

Using a well-known math lemma:

$$P_0(t) = P[N(t) = 0] = e^{-\lambda t}$$

POISSON DISTRIBUTION

After some derivations:

Poisson distribution $$P_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \ \mathrm{t} \geq 0, \ \mathrm{n} = 0,1,2,\ldots$$

(probability of *n* events occurring in the time interval]0,*t*])

Characterizes the SP {*N(t)*}, which counts event occurrences in]0,*t*], under (A1)-(A3).

Mean and Variance $$E[N(t)] = \lambda t = Var[N(t)]$$

λ Is the average *rate* at which events occur per unit time.

PROPERTIES OF THE POISSON PROCESS

Exponentially Distributed Interevent Times

Memoryless Property

Superposition of Poisson Processes

Residual Lifetime Paradox

PROPERTIES OF THE POISSON PROCESS
Exponentially Distributed Interevent Times

Q.: *How are interevent times distributed in a Poisson process*?

Let's suppose that the *(k-1)*th event takes place at some random time T_{k-1}. Let V_k denote the interevent random time following this event, with cdf $G_k(t)$

$$G_k(t) = P(V_k \le t) = 1 - P(V_k > t)$$

$$P(V_k > t \mid T_{k-1} = t_{k-1}) = P\left[N(t_{k-1}, t_{k-1} + t) = 0\right] = P\left[N(t) = 0\right] = P_0(t)$$

... But this is independent of t_{k-1}. Therefore

$$P(V_k > t \mid T_{k-1} = t_{k-1}) = P(V_k > t) = P_0(t)$$

hence

$$G_k(t) = G(t) = 1 - P_0(t) = 1 - e^{-\lambda t}, \quad t \ge 0$$

$$g(t) = \lambda e^{-\lambda t}, \quad t \ge 0$$

mean $1/\lambda$

variance $1/\lambda^2$

$P(V_k > t) = P(V > t)$ in the sequel

PROPERTIES OF THE POISSON PROCESS
Memoryless

Q.: *How are residual interevent times distributed in a Poisson process?*

$$P(V \le z+t \mid V > z) = \frac{P[(V \le z+t) \wedge (V > z)]}{P(V > z)} = 1 - e^{-\lambda t}$$

The event [*V* > *z*] has already occurred

$$\frac{P[(V \le z+t) \wedge (V > z)]}{P(V > z)} = \frac{P[(z < V \le z+t)]}{1 - P(V \le z)} = \frac{1 - e^{-\lambda(z+t)} - (1 - e^{-\lambda z})}{e^{-\lambda z}} = 1 - e^{-\lambda t}$$

This distribution is independent of the event age z and it is equal to the interevent distribution $G(t)=P(V \le t)$ ⇒ memoryless property

$$P(V \le z+t \mid V > z) = P(V \le t)$$

What about *Y*, the distribution of the *residual lifetimes*?

Note that $V = Y + z$

$$H(t,z) = P(Y \le t \mid V > z) \doteq P(V \le z+t \mid V > z) = G(t) = 1 - e^{-\lambda t}$$

Every residual lifetime in a Poisson process is characterized by the exact same exponential distribution as the original lifetime, regardless of the event age *z*.

PROPERTIES OF THE POISSON PROCESS
Memoryless

Theorem (memoryless property is unique to the exponential distribution): Let V be a positive random variable with a differentiable distribution function $G(t) = P[V \le t]$. Then:

$$H(t,z) = P[V \le z + t \mid V > z]$$

is independent of z (i.e., it is *memoryless*)

iff

$$G(t) = 1 - e^{-\lambda t}, \lambda > 0$$

And, moreover, if $G(t) = 1 - e^{-\lambda t}$, then

$$H(t,z) = 1 - e^{-\lambda t}$$

PROPERTIES OF THE POISSON PROCESS
Superposition of Poisson Processes

Considering now a DES with *m* > *1* events and assuming

- each event sequence modeled as a Poisson process with parameter λ_i, i=1,...,*m*
- these *m* Poisson processes are mutually independent

Q.: *Are the previous properties preserved in for a SP which is the superposition of* m *Poisson processes? In what form?*

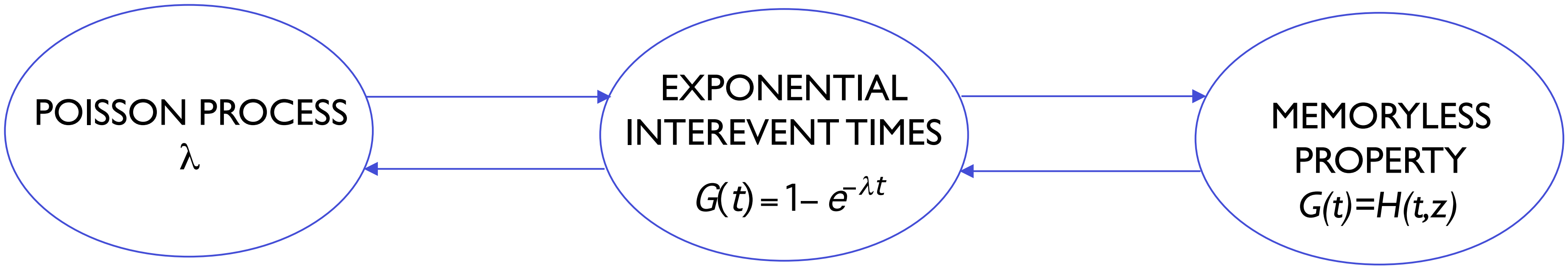
$$P[Y^* \leq t] = 1 - P[Y^* > t] = 1 - P\left[\min_i \{Y_i\} > t\right] = 1 - P[Y_1 > t \wedge Y_2 > t \wedge \ldots \wedge Y_m > t]$$

$$P[Y^* \leq t] = 1 - \prod_{i=1}^{m} P[Y_i > t]$$

The superposition of m independent Poisson processes with parameters λ_i is also a Poisson process with

$$P[Y^* \leq t] = 1 - e^{-\Lambda t}, \quad \Lambda = \sum_{i=1}^{m} \lambda_i$$

PROPERTIES OF THE POISSON PROCESS



PROPERTIES OF THE POISSON PROCESS
Residual Lifetime Paradox

- Think on a bus station where buses arrive following a Poisson distribution with parameter λ (The average time of arrival is $1/\lambda$)
- If someone arrives at a bus stop at some random time instant, how long will he/she have to wait, on the average, until the next bus?

Answer I

$1/2\lambda$ minutes

because the average time between two buses is $1/\lambda$ and he/she arrives randomly at any point in the interval

But the residual lifetime is exponentially distributed with mean $1/\lambda$, so on the average the traveller should wait $1/\lambda$

Answer II

$1/\lambda$ minutes

based on the prior argument (memoryless property)

However, the mean time since the last bus arrival is also $1/\lambda$ so the traveller should wait for $1/\lambda + 1/\lambda = 2/\lambda$.

PROPERTIES OF THE POISSON PROCESS
Residual Lifetime Paradox

Correct answer:

$2/\lambda$ because the traveller arrives at a uniformly distributed random point in the time interval, therefore he/she is more likely to fall in a large interval than in a short one.

(see formal proof on Cassandras and Lafortune, 1999)

"Randomness always increases waiting"

STA WITH POISSON CLOCK STRUCTURE

Case 1: STA with 1 single event *e* always enabled

A Poisson process with parameter λ may be viewed as a GSMP generated by an STA $(\xi,\chi,\Gamma,f,p_0,G)$

where

- $\xi=\{e\}$,
- $\chi=\{0,1,2,...\}$
- $\Gamma(x)=\xi$ for all x
- $p_0(0)=1$
- $f(x,e)=x+1$ for all x (in this case the transitions have all probability 1)
- *clock structure* $G(t)=1-e^{-\lambda t}, \lambda > 0$

STA WITH POISSON CLOCK STRUCTURE

Case 2: STA with *m* events and no constraints on $\Gamma(x)$ and *p* (transitions may be deterministic or not)

A Poisson process may be viewed as a GSMP generated by an STA $(\xi,\chi,\Gamma,p,p_0,G)$ where

- $\xi=\{e_1,\ldots,e_m\}$,
- $\chi=\{0,1,2,\ldots\}$
- $\Gamma(x)$ has no constraints (some events may be disabled in certain states)
- p_0 has no constraints
- $p(x'|x,e')$ has no constraints (not all transitions have necessarily probability 1)
- *clock structure* $G=\{G_i, i=1,\ldots,m\}$ $\quad G_i(t)=1-e^{-\lambda_i t}, \lambda_i>0$ (*m* independent Poisson processes)

STA WITH POISSON CLOCK STRUCTURE

Distribution of Interevent Times at state x:

$$G(t,x) = P\left[Y^*(x) \le t\right] = P\left[\min_{i \in \Gamma(x)} \{Y_i\} \le t\right] = 1 - e^{-\Lambda(x)t}$$

$$\Lambda(x) = \sum_{i \in \Gamma(x)} \lambda_i$$

This is a generalization of $P[Y^* \le t] = 1 - e^{-\Lambda t}, \quad \Lambda = \sum_{i=1}^{m} \lambda_i$ where we assumed $\Gamma(x) = \xi$ for all x

Intuitively, if some event is occasionally disabled, when it is reenabled, its lifetime is simply taken to be some residual lifetime from the original Poisson clock sequence. This residual lifetime has the same distribution as a lifetime (memoryless property). The only effect of enabling and disabling events is that the Poisson parameter $\Lambda(x)$ depends on the state. Therefore, interevent times are no longer identically distributed.

STA WITH POISSON CLOCK STRUCTURE

Distribution of Triggering Event for a given state x:

$$p(i \mid x) = P\left[E' = i \mid X = x\right], \quad i = 1, \ldots, m$$

$$p(i \mid x) = \frac{\lambda_i}{\Lambda(x)}$$

[Derivation in Cassandras & Lafortune book]

Depends on the current state through $\Lambda(x) = \sum_{i \in \Gamma(x)} \lambda_i$

STA WITH POISSON CLOCK STRUCTURE

A GSMP with a Poisson clock structure is a Markov Chain with

$$p(x'|x) = \sum_{i \in \Gamma(x)} P[X' = x' | E' = i, X = x].P[E' = i | X = x]$$

$$p(x'|x) = \sum_{i \in \Gamma(x)} p(x'|x,i) \frac{\lambda_i}{\Lambda(x)}$$

part of the GSMP definition (only depends on *x*)

only depends on *x*

STA WITH POISSON CLOCK STRUCTURE

Generation of a sample path of a Markov Chain:

Given $G = \{\lambda_1,\ldots, \lambda_m\}$:

Step 1: With x known, evaluate the feasible event set $\Gamma(x)$.

Step 2: For every event $i \in \Gamma(x)$, sample from $G_i(t) = 1 - e^{-\lambda_i t}$ to obtain a clock value y_i.

Step 3: The triggering event is $e' = \arg\min_{i \in \Gamma(x)} \{y_i\}$.

Step 4: The next state x' is obtained by sampling from $p(x'|x,e')$.

Step 5: The next event time is given by $t' = t + \min_{i \in \Gamma(x)} \{y_i\}$.

STOCHASTIC PETRI NETS

Def.:A *Stochastic PN* is a 6-tuple $(P,T,A,w,x,\mathbf{F})$ where (P,T,A,w,x) is a marked PN, and $\mathbf{F}: R[x_0] \times T \rightarrow \Re$ is a function that associates to each transition t in each reachable marking x a random variable

Def.:A *Generalized Stochastic PN* is a 7-tuple $(P,T=T_0 \cup T_D,A,w,x,\mathbf{F},\mathbf{S})$ where (P,T,A,w,x) is a marked PN, $\mathbf{F}: R[x_0] \times T_D \rightarrow \Re$ is a function that associates to each *timed* transition $t \in T_D$ in each reachable marking x a random variable. Each $t \in T_0$ has zero firing time in all reachable x.

$\mathbf{S}$ is a set (possibly empty) of elements called *random switches*, which associate probability distributions to subsets of conflicting immediate transitions.

EXPONENTIAL TIMED PETRI NETS

For **Exponential Timed PN**s, in the two previous definitions **F:**$R[x_0] \times T \rightarrow \Re$ is a function that associates to each transition $t_j \in T_D$ in each reachable marking x an *exponential* random variable with rate $\lambda_j(x)$.

The transitions in T_D are known as *exponential transitions* and refer to $\lambda_j(x)$ as the *firing rate* of t_j in x.

EXPONENTIAL TIMED PETRI NETS

Theorem – The marking process of an exponential timed Petri net is a continuous time Markov Chain (CTMC).

State space of the equivalent CTMC: reachability set $R[x_0]$ of the exponential timed Petri net

Computation of the transition rate from state x_i to state $x_j \neq x_i$ is given by

$$q_{ij} = \sum_{t_k \in T_{ij}} \lambda_k(x_i)$$

Where T_{ij} is the subset of T_D of enabled transitions in x_i such that the firing of any transition in T_{ij} leaves the CTMC in x_j .

If $x_j = x_i$,

$$q_{ii} = -\sum_{j \neq i} q_{ij}$$

EXPONENTIAL TIMED PETRI NETS

When there is *conflict* in state x_i , if T_i is the set of enabled transitions in x_i, the probability of firing $t_j \in T_i$ is:

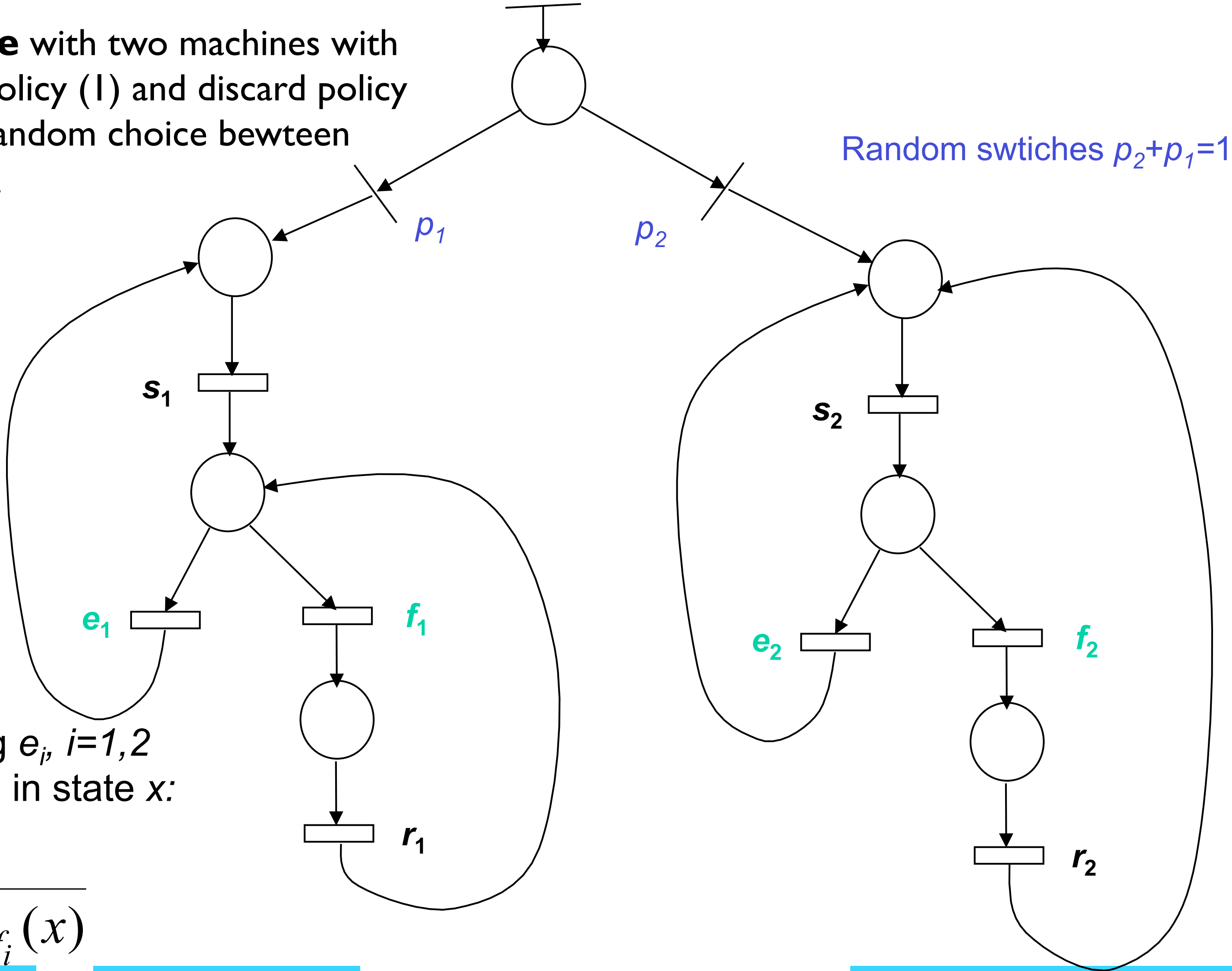
- if T_i is composed by exponential transitions only:

$$\frac{\lambda_j(x_i)}{\sum_{t_k \in T_i} \lambda_k(x_i)}$$

- if T_i includes one single immediate transition, this is the one that will fire
- if T_i includes two or more immediate transition, a probability mass function will be specified over them by an element of *S*. The subset of immediate transitions plus the switching distribution is called a *random switch*.

EXPONENTIAL TIMED PETRI NETS

Example with two machines with resume policy (1) and discard policy (2), and random choice bewteen machines.



Probability of firing e_i, $i=1,2$ instead of f_i, $i=1,2$, in state x:

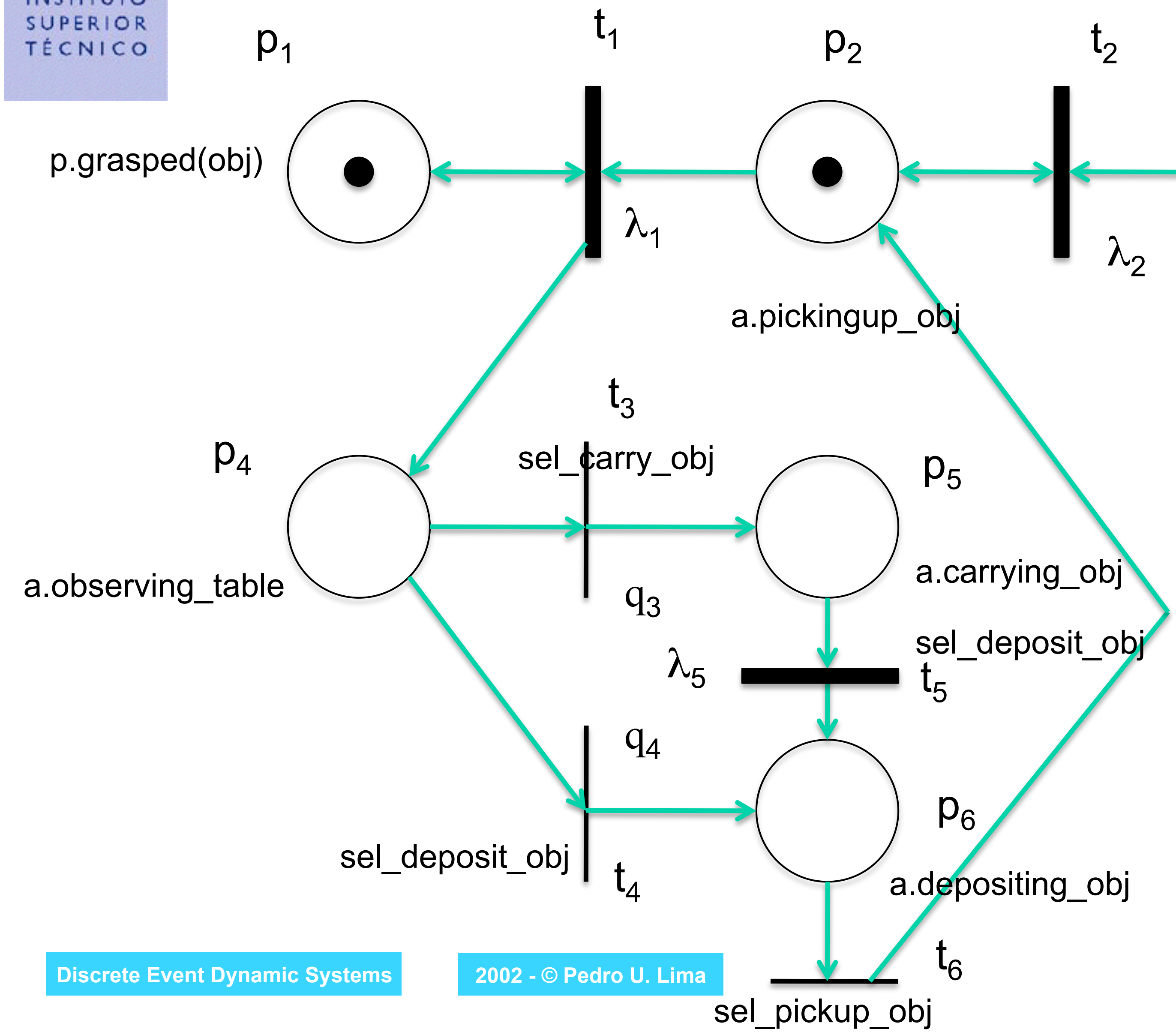
$$\frac{\lambda_{e_i}(x)}{\lambda_{e_i}(x)+\lambda_{f_i}(x)}$$

GSPN AND EQUIVALENT CTMC

To ensure the existence of an unique steady state probability vector $(\rho_1,\ldots,\rho_s)$ for the marking process of the GSPN with *s* tangible markings, the following simplifying assumptions are made:

1. The GSPN is bounded, i.e., its reachability set is finite
2. Firing rates do not depend on time parameters, ensuring that the equivalent MC is homogeneous
3. The GSPN model is proper and deadlock-free, i.e., the initial marking is reachable with a non-zero probability from any marking in the reachability set and also there is no absorbing marking (*can be lifted*)

EXAMPLE: GSPN AND EQUIVALENT CTMC

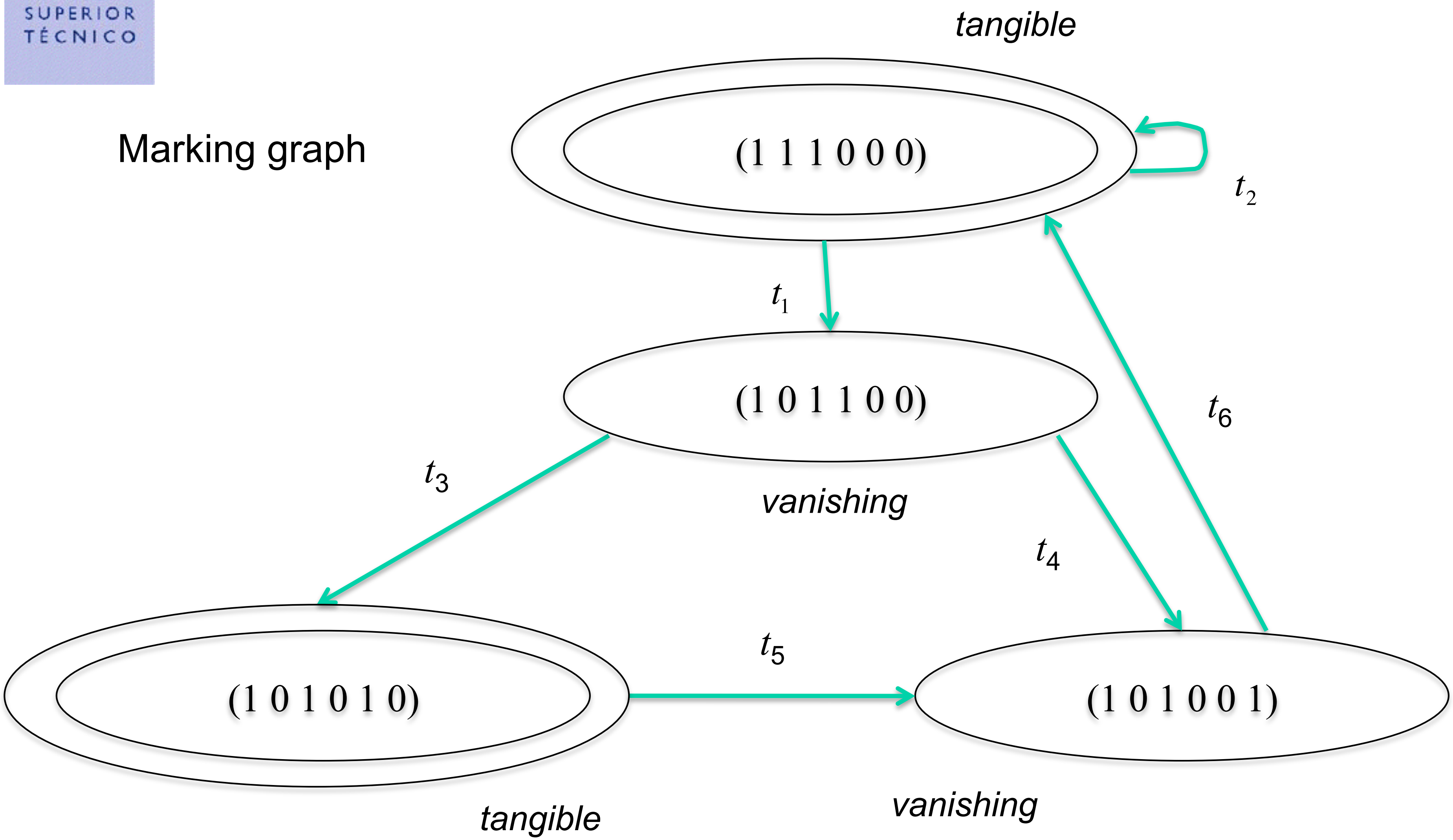


$q_3 + q_4 = 1$

random switches

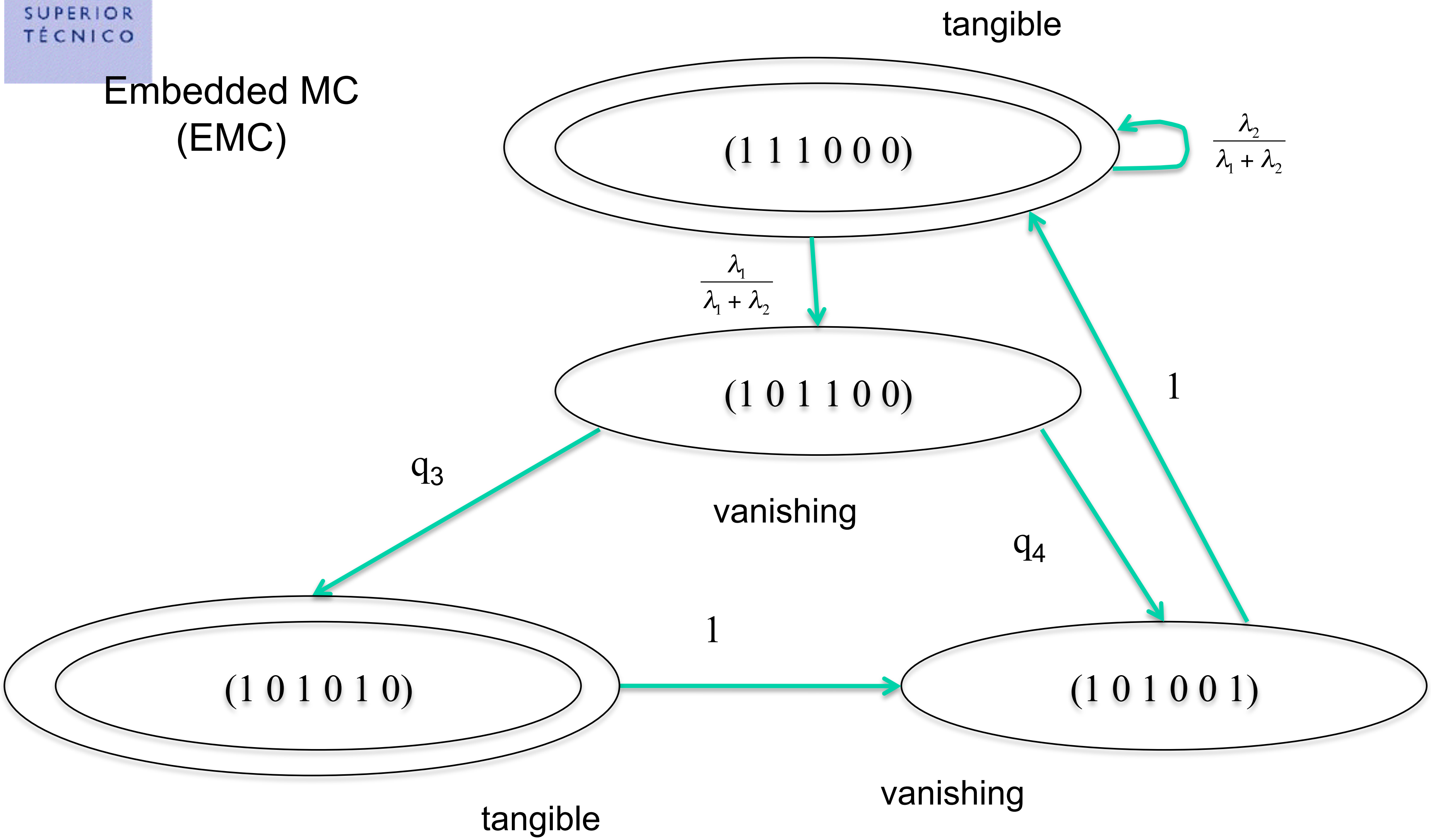
EXAMPLE: GSPN AND EQUIVALENT CTMC

Marking graph



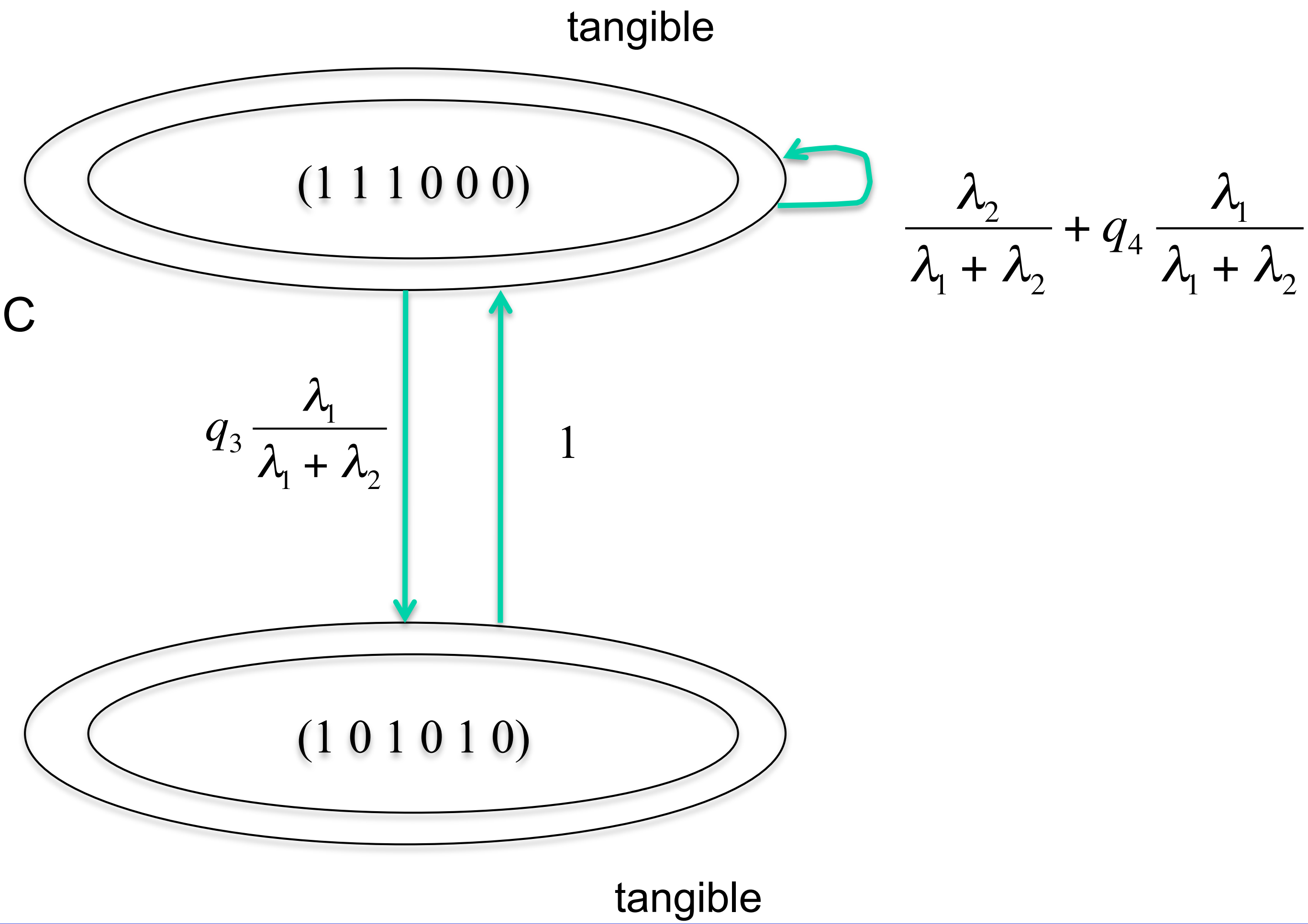
EXAMPLE: GSPN AND EQUIVALENT CTMC

Embedded MC (EMC)



EXAMPLE: GSPN AND EQUIVALENT CTMC

Reduced Embedded MC
(REMC)



MDP: random switch probabilities can be manipulated to achieve optimal decision

GSPN, REMC AND PERFORMANCE MEASURES

PNs of robot controller and world model must be connected in closed loop. Closed loop PN can be analyzed w.r.t., e.g.,

1. Probability that a particular condition *C* holds

$$\Pr(C) = \sum_{j \in S_1} \rho_j, \quad S_1 = \left\{ j \in \{1,\ldots,s\} : C \text{ is satisfied in } \mathbf{x}_j \right\}$$

2. Probability that place p_i has exactly *k* tokens

$$\Pr(p_i,k) = \sum_{j \in S_2} \rho_j, \quad S_2 = \left\{ j \in \{1,\ldots,s\} : \mathbf{x}_j(p_i) = k \right\}$$

3. Expected number of tokens in a place p_i:

$$ET[p_i] = \sum_{k=1}^{K} k \Pr(p_i,k),$$

where K is the max number of tokens p_i may contain in any reachable marking

GSPN, REMC AND PERFORMANCE MEASURES cont'd

5. Throughput rate of an exponential transition t_j:

$$TR(t_j) = \sum_{i \in S_3} \rho_i \lambda(\mathbf{x}_i, t_j) \upsilon_{ij}, \quad S_3 = \left\{ i \in \{1, \ldots, s\} : t_j \text{ enabled in } \mathbf{x}_i \right\}$$

where υ_{ij} is the probability that t_j fires among all enabled transitions in $\mathbf{x}_i$

6. Throughput rate of immediate transitions can be computed from those of the exponential transitions and from the structure of the model

7. Mean waiting time in a place p_i:

$$WAIT(p_i) = \frac{ET[p_i]}{\sum_{t_j \in IN(p_i)} TR(t_j)} = \frac{ET[p_i]}{\sum_{t_j \in OUT(p_i)} TR(t_j)}$$

STOCHASTIC TIMED AUTOMATA

Further reading

- Extensions of the GMSP
- Falko Bause, Pieter S Kritzinger, *Stochastic Petri Nets - An Introduction to the Theory,* Vieweg Advanced Studies in Computer Science, 2002 – pdf version
- N. Viswanadham, Y. Narahari, *Performance Modeling of Automated Manufacturing Systems*,Prentice-Hall, 1992 - more on modeling of manufacturing systems

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